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ON THE MATRIX EQUATIONS  $AH + HA^* = A^*H + HA = I$

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ON THE MATRIX EQUATIONS  $AH + HA^* = A^*H + HA = I$

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This paper considers the simultaneous solution of the matrix equations  $AH + HA^* = A^*H + HA = I$  where  $H$  is Hermitian. A full characterization of  $A$  in terms of a given  $H$  is obtained. Various results are also obtained for those  $H$  satisfying the equations with a given  $A$ .

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## I. Introduction

We consider the problem of finding those matrices  $A$  and Hermitian matrices  $H$  which simultaneously satisfy

$$AH + HA^* = I$$

and

(1)

$$A^*H + HA = I.$$

This problem was suggested by Taussky [3]. Davis [2] obtained some results for normal or real matrices  $A$ . Partial results were also obtained by Barker [1] who was mostly concerned with ascertaining conditions such that an  $A$  which satisfies (1) is normal. In this work we obtain more general results which include some results of Davis and Barker as special cases. In one instance, a result of Barker is proved under a less stringent hypothesis.

We shall find it convenient to consider

$$(A+A^*)H + H(A+A^*) = 2I$$

(2)

and

$$(A-A^*)H = H(A-A^*)$$

(3)

which are equivalent to (1). Then from either of (1) and (2) it is well known, e.g. Taussky [4], that  $A$ ,  $H$  and  $(A+A^*)$  must be nonsingular. Also (3) obviously states that  $H$  and the skew-Hermitian part of  $A$  must commute if (1) is to hold.

## II. Characterization of $A$ for Given $H$

Suppose that  $U$  is a unitary matrix which diagonalizes  $H$ , i.e.

$$U^*HU = G = \text{diag}(\gamma_1, \dots, \gamma_n)$$

(4)

where  $\gamma_j$  are the eigenvalues of  $H$ . Then define

$$B = U^*(A+A^*)U$$

and

(5)

$$S = U^*(A-A^*)U$$

so that (2) and (3) become

$$BG + GB = 2I \quad (6)$$

and

$$SG = GS. \quad (7)$$

For a given matrix  $H$  we may completely characterize the matrix  $A$  by:

**THEOREM 1.** Let  $\gamma_j$ ,  $j = 1, \dots, n$ , be the eigenvalues and  $U$  a unitary diagonalizer of a given Hermitian matrix  $H$ . Then the Hermitian part of a matrix  $A$  which satisfies (1) is given by

$$A + A^* = H^{-1} + UB_1U^* \quad (8)$$

where  $B_1$  is a Hermitian matrix whose entries are zero except when  $\gamma_i + \gamma_j = 0$ , in which case the  $(i,j)$  entry of  $B_1$  is arbitrary. Furthermore, the skew-Hermitian part of  $A$  is given by

$$A - A^* = US_1U^* \quad (9)$$

where  $S_1$  is a skew-Hermitian matrix whose diagonal entries are arbitrary imaginary numbers and whose  $(i,j)$ th entry is zero whenever  $\gamma_i \neq \gamma_j$  and is arbitrary otherwise.

**Proof:** If  $b_{ij}$  is an element of the Hermitian matrix  $B$  defined by (5), then (6) yields

$$b_{ij}(\gamma_i + \gamma_j) = 2\delta_{ij} \quad i, j = 1, \dots, n.$$

Since  $H$  is nonsingular,  $b_{ii} = 1/\gamma_i$ ,  $b_{ij} = 0$  if  $(\gamma_i + \gamma_j) \neq 0$ , and  $b_{ij} = b_{ji}$  is arbitrary whenever  $(\gamma_i + \gamma_j) = 0$ . Then  $B = G^{-1} + B_1$ , which using (4) and (5) results in (8). If  $s_{ij}$  is an element of the skew-Hermitian matrix  $S$ , then (7) can be expressed as

$$s_{k\ell}(\gamma_k - \gamma_\ell) = 0 \quad k, \ell = 1, \dots, n$$

so that  $s_{kk}$  is arbitrary,  $s_{k\ell} = 0$  whenever  $\gamma_k \neq \gamma_\ell$ , and  $s_{k\ell}$  is arbitrary whenever  $\gamma_k = \gamma_\ell$ . Thus, with  $S_1 = S$ , (5) results in (9). ■

A consequence of Theorem 1 is:

**COROLLARY 1.1.** *If  $(\gamma_i + \gamma_j) \neq 0$  for all  $i$  and  $j$  and if (1) holds, then  $A$  and  $H$  commute and  $A$  is normal.*

*Proof:* If  $(\gamma_i + \gamma_j) \neq 0$  for all  $i$  and  $j$ , then  $B_1 = 0$  and  $(A+A^*) = H^{-1}$  so that  $(A+A^*)$  trivially commutes with  $H$ . But from (3),  $(A-A^*)$  also commutes with  $H$  so that  $A$  and  $H$  commute. Furthermore, if  $(A-A^*)$  commutes with  $H$ , it also commutes with  $H^{-1}$ , so that  $(A-A^*)$  commutes with  $(A+A^*)$  and therefore  $A$  is normal. ■

This corollary was proven by Barker [1] with the additional (and unnecessary) assumption of  $\gamma_i \neq \gamma_j$ . Furthermore, the proof of the corollary contains the proof of the following theorem, which is also contained in Barker:

**THEOREM 2.** *If (1) holds for some Hermitian matrix  $H$  and if  $(A+A^*)$  and  $H$  commute, then  $A$  is normal.*

Then we also have:

**COROLLARY 2.1.** *If (1) holds for some Hermitian matrix  $H$  and if  $A$  and  $H$  commute, then  $A$  is normal.*

*Proof:* If  $A$  and  $H$  commute, then so do  $(A+A^*)$  and  $H$ , so that by Theorem 2,  $A$  is normal. ■

### III. Characterization of H for Given A

We now seek Hermitian matrices  $H$  such that (1) is satisfied for a given complex matrix  $A$ . We first characterize such  $H$  using the diagonalizer of the skew-Hermitian part of the given matrix  $A$ :

**THEOREM 3.** *Let  $V$  be a unitary diagonalizer of the skew-Hermitian part of a given matrix  $A$ . Let  $B = V^*(A+A^*)V$ . Then if the  $(i,j)$  element  $b_{ij}$  of  $B$  is zero whenever  $(b_{ii}+b_{jj}) \neq 0$ ,  $H$  given by*

$$H = V \operatorname{diag}(b_{11}^{-1}, \dots, b_{nn}^{-1}) V^* \quad (10)$$

*is a solution of (1).*

*Proof:* With  $H$  given by (10), (2) can be expressed as

$$b_{ij} (1/b_{ii} + 1/b_{jj}) = 2\delta_{ij} \quad i, j = 1, \dots, n \quad (11)$$

which is an identity for  $i = j$ . For  $i \neq j$ , (11) becomes

$$b_{ij} (b_{ii} + b_{jj}) = 0$$

which is again an identity due to the hypothesis of the theorem. Furthermore, (3) holds trivially, since under the unitary transformation  $V$ ,  $H$  and  $(A-A^*)$  are both diagonalized. Since (2) and (3) hold, then so does (1). ■

*Remark:* If the eigenvalues of  $(A-A^*)$  are distinct, then the hypothesis about the elements of  $B$  are necessary for a solution  $H$  of (1) to exist. Furthermore, if such a solution does exist, it is uniquely given by (10).

We now consider the case where  $(A+A^*)$  and  $-(A+A^*)$  have no common eigenvalue:

THEOREM 4. Let  $\beta_i, i = 1, \dots, n$ , be the eigenvalues and  $W$  be a unitary diagonalizer of the Hermitian part of a given matrix  $A$ . Let  $S = W(A - A^*)W$ . Then if  $(\beta_i + \beta_j) \neq 0$  for all  $i$  and  $j$ , (1) has a solution  $H$  only if  $s_{ij} = 0$  whenever  $\beta_i \neq \beta_j$ . If this is the case, then  $H = (A + A^*)^{-1}$ .

*Proof:* Let

$$G = W^*HW. \quad (12)$$

Then (2) can be expressed as

$$g_{ij}(\beta_i + \beta_j) = 2\delta_{ij} \quad i, j = 1, \dots, n$$

where  $g_{ij}$  is an element of  $G$ , so that

$$g_{ii} = 1/\beta_i \quad \text{and} \quad g_{ij} = 0, \quad i \neq j. \quad (13)$$

Then (3) can be written as

$$s_{ij}(g_{ii} - g_{jj}) = 0 \quad i, j = 1, \dots, n$$

or from (13)

$$s_{ij}(\beta_i - \beta_j) = 0 \quad i, j = 1, \dots, n \quad (14)$$

where  $s_{ij}$  is an element of  $S$ . Therefore either  $s_{ij} = 0$  or  $\beta_i = \beta_j$  for a solution to exist. Then if (14) is valid for all  $i$  and  $j$ , (12) and (13) yield  $H = (A + A^*)^{-1}$ . ■

Another result of Barker follows immediately from this theorem:

COROLLARY 4.1. If (1) holds for some Hermitian matrix  $H$  and if  $(A + A^*)$  and  $-(A + A^*)$  have no common eigenvalue, then  $A$  is normal.

*Proof:* By Theorem 4, if  $(A + A^*)$  and  $-(A + A^*)$  have no common eigenvalue and (1) holds, then  $H = (A + A^*)^{-1}$  so that trivially  $(A + A^*)$  and  $H$  commute. Then, by Theorem 2,  $A$  is normal. ■

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